

FORMALITY CONJECTURES FOR CHAINS

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1. INTRODUCTION

In [K], Kontsevich proved a classification theorem for deformation quantizations of $C^\infty(M)$ where M is a smooth manifold. This theorem asserts that the set of isomorphism classes of deformations of $C^\infty(M)$ is in one-to-one correspondence with the set of equivalence classes of formal Poisson structures on M . This theorem follows from a more general theorem: the differential graded algebra $\mathfrak{g}_S^\bullet(M)$ of multi-vector fields on M is equivalent to the differential graded algebra $\mathfrak{g}_G^\bullet(C^\infty(M))$ of Hochschild cochains of $C^\infty(M)$ (we recall exact definitions and statements from [K] and references thereof in section 2). In other words, the algebra of Hochschild cochains is *formal* (equivalent to its cohomology).

In this paper, we state a formality conjecture about the Hochschild and cyclic chain complexes of the algebra $C^\infty(M)$. It is well known that for any algebra A the Hochschild chain complex $C_\bullet(A, A)$ and the negative cyclic complex $CC_\bullet^-(A)$ are modules over the Lie algebra of Hochschild cochains $\mathfrak{g}_G^\bullet(A)$. Therefore, by virtue of the Kontsevich formality theorem, both the Hochschild (resp. negative cyclic) complex and the graded space of differential forms (resp. de Rham complex) are strong homotopy modules over $\mathfrak{g}_G^\bullet(C^\infty(M))$. We conjecture that those modules are equivalent in an appropriate sense (cf. section 3), or, in other words, that Connes' quasi-isomorphism from [C], [L] is, in the right sense, $\mathfrak{g}_G^\bullet(C^\infty(M))$ -equivariant. As in [K], the correct language for stating our conjectures is that of homotopical algebra of Stasheff.

We derive several consequences from the above conjecture in section 4. First, we compute the Hochschild and cyclic homology of deformed algebras given by the classification theorem of Kontsevich. In particular, we compute the space of traces on such a deformed algebra. Then, in subsection 4.1, we show how to construct the \hat{A} class of an arbitrary Poisson manifold. In the case of a regular Poisson structure, this class is, conjecturally, the \hat{A} class of the tangent bundle to the foliation of symplectic leaves.

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Finally, in section 5 we outline a possible proof of our conjectures, as well as their generalization, along the lines of a recent work of Tamarkin [T].

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2. FORMALITY THEOREM OF KONTSEVICH

2.1. Classification of star products. Let A_0 be an associative unital algebra over a commutative unital ring k . A *deformation* [G] of A_0 is a formal power series

$$a * b = ab + \sum_{m=1}^{\infty} t^m P_m(a, b)$$

where $P_m : A_0 \times A_0 \rightarrow A_0$ are k -bilinear forms such that

- a) The product $*$ is associative
- b) $1 * f = f * 1 = f$, $f \in A_0$.

An *isomorphism* of two deformations $*$, $*$ ' is a formal power series $T(a) = a + \sum_{m=1}^{\infty} t^m T_m(a)$ such $T(a * b) = T(a) *' T(b)$ for a, b in A_0 .

Let M be a C^∞ manifold. A *deformation quantization* of $C^\infty(M)$, or a *star product*, is a deformation of $A_0 = C^\infty(M)$ such that P_m are bidifferential operators [BFFLS]. An *isomorphism* of two star products is an isomorphism of corresponding deformations such that the operators T_m are differential.

Given a star product on a C^∞ manifold M , one defines a Poisson bracket on $C^\infty(M)$ by

$$\{f, g\} = P_1(f, g) - P_1(g, f) \quad (2.1)$$

For a bivector field π , put

$$\{f, g\}_\pi = \langle \pi, df \wedge dg \rangle \quad (2.2)$$

It follows from associativity of $*$ modulo t^2 that the Poisson bracket (2.1) is necessarily of the form $\{f, g\}_{\pi_0}$ for some bivector field π_0 .

Recall that for a bivector field π there exists unique trivector field $[\pi, \pi]_S$ such that

$$\{f, \{g, h\}_\pi\}_\pi + \{g, \{h, f\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi = \langle [\pi, \pi]_S, df \wedge dg \wedge dh \rangle \quad (2.3)$$

for any smooth functions f, g , and h . The expression $[\pi, \pi]_S$ is quadratic in π . The polarization of $[\pi, \pi]_S$ is a symmetric bilinear form $[\pi, \psi]_S$ with values in the space $\Gamma(M, \wedge^3 T)$ of trivector fields. A bivector field π such that $[\pi, \pi]_S = 0$ is by definition a *Poisson structure*

on M . It follows from associativity of $*$ modulo t^3 that for any star product $*$, the bivector field π_0 (formula (2.2)) is a Poisson structure.

A *formal Poisson structure* is by definition a formal power series $\pi = \sum_{m=0}^{\infty} t^{m+1} \pi_m$ such that $[\pi, \pi]_S = 0$ in $\Gamma(M, \wedge^3 T)[[t]]$. Every Poisson structure π defines a formal Poisson structure $t\pi$. Two formal Poisson structures π and π' are *equivalent* if there is a formal power series $X = \sum_{m=1}^{\infty} t^m X_m$ such that $\pi' = \exp(L_X)\pi$.

Theorem 2.1.1. (*Kontsevich, [K]*). *There is a bijection, natural with respect to diffeomorphisms, between the set of equivalence classes of formal Poisson structures on M and the set of isomorphism classes of deformation quantizations of $C^\infty(M)$.*

If $\pi = \sum_{m=0}^{\infty} t^{m+1} \pi_m$ is a formal Poisson structure, we will denote by $*_\pi$ a star product from the equivalence class corresponding to π by the above theorem. The Poisson structure associated to $*_\pi$ by formulas (2.1) and (2.2) will then be equal to π_0 .

2.2. Hochschild cochains. For a unital algebra A over a commutative unital ring k , for $n \geq 0$ let

$$\tilde{C}^n(A, A) = \text{Hom}(A^{\otimes n}, A) \quad (2.4)$$

If $A = C^\infty(M)$, we require that $\tilde{C}^n(A, A)$ consist of those maps from $A^{\otimes n}$ to A which are multi-differential.

Define the *Gerstenhaber bracket* (cf. [G])

$$[\ , \]_S : \tilde{C}^m(A, A) \otimes \tilde{C}^m(A, A) \rightarrow \tilde{C}^{n+m-1}(A, A)$$

as follows. For $D \in \tilde{C}^m(A, A)$ and $E \in \tilde{C}^m(A, A)$ put

$$\begin{aligned} (D \circ E)(a_1, \dots, a_{n+m-1}) = \\ = \sum_{j=0}^{n-1} (-1)^{(m-1)j} D(a_1, \dots, a_j, E(a_{j+1}, \dots, a_{j+m}), \dots); \end{aligned} \quad (2.5)$$

$$[D, E]_S = D \circ E - (-1)^{(n-1)(m-1)} E \circ D \quad (2.6)$$

The bracket $[\ , \]_S$ turns $\tilde{C}^{\bullet+1}(A, A)$ into a graded Lie algebra ([G]). Put

$$m(a, b) = ab \quad (2.7)$$

for $a, b \in A$. One has $[m, m] = 0$ (this is equivalent to m being associative), so the operator

$$\delta = [m, ?] : \tilde{C}^\bullet(A, A) \rightarrow \tilde{C}^{\bullet+1}(A, A) \quad (2.8)$$

satisfies $\delta^2 = 0$. The complex $(\tilde{C}^\bullet(A, A), \delta)$ is called the unnormalized Hochschild cochain complex of A with coefficients in the bimodule A . The cohomology of this complex is denoted by $H^\bullet(A, A)$ (the Hochschild cohomology).

Define the (normalized) *Hochschild cochain complex* of A with coefficients in A by

$$C^n(A, A) = \text{Hom}(\overline{A}^{\otimes n}, A) \quad (2.9)$$

where $\overline{A} = A/k1$. It is easy to see that $C^\bullet(A, A)$ is a subcomplex and a graded Lie subalgebra of $\tilde{C}^\bullet(A, A)$. It is well known that the embedding of $C^\bullet(A, A)$ into $\tilde{C}^\bullet(A, A)$ is a quasi-isomorphism ([CE]).

Put

$$\mathfrak{g}_G^\bullet(A) = C^{\bullet+1}(A, A) \quad (2.10)$$

The differential δ and the bracket $[\cdot, \cdot]_S$ make $\mathfrak{g}_G^\bullet(A)$ a differential graded Lie algebra.

Remark 2.2.1. Formula

$$\begin{aligned} (D \smile E)(a_1, \dots, a_{n+m}) &= \\ &= (-1)^{nm} D(a_1, \dots, a_n) E(a_{n+1}, \dots, a_{n+m}) \end{aligned} \quad (2.11)$$

defines an associative product on $C^\bullet(A, A)$. This product is compatible with the differential δ , therefore $C^\bullet(A, A)$ is a differential graded algebra.

The following theorem is essentially contained in [HKR]

Theorem 2.2.2. *The formula*

$$D_\pi(a_1, \dots, a_n) = \langle \pi, da_1 \dots da_n \rangle$$

defines a quasi-isomorphism

$$(\Gamma(T, \wedge^\bullet T), 0) \rightarrow C^\bullet(C^\infty(M), C^\infty(M))$$

Under the isomorphism

$$\Gamma(T, \wedge^\bullet T) \rightarrow H^\bullet(C^\infty(M), C^\infty(M))$$

which is induced by this map on cohomology, the bracket induced by $[\cdot, \cdot]_G$ becomes the Schouten bracket $[\cdot, \cdot]_S$ (2.17) and the product induced by the cup product becomes the wedge product.

2.3. L_∞ algebras. An L_∞ algebra is a graded vector space \mathfrak{g}^\bullet together with a coderivation ∂ of the free cocommutative coalgebra $S(\mathfrak{g}^\bullet[1])$ such that degree of ∂ is 1 and $\partial^2 = 0$.

Put

$$\wedge^m(\mathfrak{g}^\bullet) = S^m(\mathfrak{g}^\bullet[1])[-m] \quad (2.12)$$

One has

$$\wedge^\bullet(\mathfrak{g}^\bullet) = T(\mathfrak{g}^\bullet)/I \quad (2.13)$$

where I is the two-sided ideal generated by all the elements

$$xy - (-1)^{(|x|-1)(|y|-1)}yx$$

where x, y are homogeneous elements of \mathfrak{g}^\bullet . A coderivation ∂ of degree 1 is uniquely determined by its composition with the projection $S(\mathfrak{g}^\bullet[1]) \rightarrow \mathfrak{g}^\bullet[1]$ which is a sequence of maps

$$[\dots]_n : \wedge^n \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^\bullet \quad (2.14)$$

of degree $2 - n$, $n = 1, 2, \dots$. The condition $\partial^2 = 0$ is equivalent to the following: for any homogeneous elements x_1, \dots, x_n of \mathfrak{g}^\bullet ,

$$\sum \pm [[x_{i_1}, \dots, x_{i_p}]_p, x_{j_1}, \dots, x_{j_q}]_{q+1} = 0 \quad (2.15)$$

where the sum is taken over all $i_1 < \dots < i_p, j_1 < \dots < j_q$ such that $\{1, \dots, n\}$ is the disjoint union of $\{i_1, \dots, i_p\}$ and $\{j_1, \dots, j_q\}$. The signs \pm are computed by the following rule: whenever a transposition of x and y appears, the result is multiplied by the sign $(-1)^{(|x|-1)(|y|-1)}$.

In particular, $\delta x = [x]_1$ is a differential on \mathfrak{g}^\bullet and the bracket $[\ , \]_2$ induces a graded Lie algebra structure on the cohomology of the complex $(\mathfrak{g}^\bullet, \delta)$.

Any differential graded algebra $(\mathfrak{g}^\bullet, [\ , \], \delta)$ is an L_∞ algebra if one puts $[\]_1 = \delta$, $[\ , \]_2 = [\ , \]$ and $[\dots]_n = 0$ for $n > 2$.

Given two L_∞ algebras \mathfrak{g}^\bullet and \mathfrak{h}^\bullet , an L_∞ morphism $f : \mathfrak{g}^\bullet \rightarrow \mathfrak{h}^\bullet$ is a morphism of differential graded coalgebras $S(\mathfrak{g}^\bullet[1]) \rightarrow S(\mathfrak{h}^\bullet[1])$. A morphism of graded coalgebras, without assuming that it commutes with differentials, is uniquely determined by its composition with the projection $S(\mathfrak{h}^\bullet[1]) \rightarrow \mathfrak{h}^\bullet[1]$, which is a sequence of linear maps $f_n : \wedge^n \mathfrak{g}^\bullet \rightarrow \mathfrak{h}^\bullet$ of degree $1 - n$. The condition that these maps define a morphism of differential coalgebras is equivalent to the following: for any homogeneous elements x_1, \dots, x_n of \mathfrak{g}^\bullet ,

$$\begin{aligned} & \sum \pm f_{q+1}([x_{i_1}, \dots, x_{i_p}]_p, x_{j_1}, \dots, x_{j_q}) = \\ & \sum \pm \frac{1}{k!} [f_{n_1}(x_{i_{11}}, \dots, x_{i_{1n_1}}), \dots, f_{n_k}(x_{i_{k1}}, \dots, x_{i_{kn_k}})]_k \end{aligned} \quad (2.16)$$

The signs \pm in (2.16), and in the sum in the left hand side, are as in (2.14). The sum in the right hand side is taken over all $k \geq 1$ and all $i_{r1} < \dots < i_{rn_r}$, $1 \leq r \leq k$, such that $\{1, \dots, n\}$ is a disjoint union of $\{i_{r1}, \dots, i_{rn_r}\}$, $1 \leq r \leq k$.

In particular, f_1 is a morphism of complexes. We say that f is an L_∞ quasi-isomorphism if f_1 is a quasi-isomorphism.

2.4. Formality theorem. Recall that for a manifold M one can define the Schouten-Nijenhuis bracket

$$[\ , \]_S : \Gamma(M, \wedge^n T) \otimes \Gamma(M, \wedge^m T) \rightarrow \Gamma(M, \wedge^{n+m-1} T) \quad (2.17)$$

as the unique bilinear operation satisfying the following conditions:

1. for $X \in \Gamma(M, T)$, $[X, \pi]_S = L_X \pi$
2. for $f, g \in \Gamma(M, \wedge^0 T)$, $[f, g]_S = 0$
3. the bracket $[\ , \]_S$ turns $\Gamma(M, \wedge^{\bullet+1} T)$ into a graded Lie algebra
4. for any $\pi, \varphi, \psi \in \Gamma(M, \wedge^{\bullet} T)$,

$$[\pi, \varphi \wedge \psi]_S = [\pi, \varphi]_S \wedge \psi + (-1)^{|\pi|(|\varphi|+1)} \varphi \wedge [\pi, \psi]_S$$

(for $\pi \in \Gamma(M, \wedge^n T)$, we write $|\pi| = n - 1$). When $n = m = 2$, the above bracket coincides with the polarized bracket from (2.3).

Denote by $\mathfrak{g}_S^\bullet(M)$ the differential graded Lie algebra $\Gamma(M, \wedge^{\bullet+1} T)$ with the bracket $[\ , \]_S$ and the differential $\delta = 0$.

Theorem 2.4.1. (*Kontsevich*, [K]). *There exists natural L_∞ quasi-isomorphism*

$$K : \mathfrak{g}_S^\bullet(M) \rightarrow \mathfrak{g}_G^\bullet(C^\infty(M))$$

The component K_1 of K coincides with the quasi-isomorphism from Theorem 2.2.2

Proof of Theorem 2.1.1. For any differential graded Lie algebra $(\mathfrak{g}^\bullet, [\ , \], \delta)$, define

$$MC(\mathfrak{g}^\bullet) = \{\pi \in t\mathfrak{g}^1[[t]] \mid \delta\pi + \frac{1}{2}[\pi, \pi] = 0\} \quad (2.18)$$

The group $\exp(t\mathfrak{g}^0[[t]])$ acts on $MC(\mathfrak{g}^\bullet)$ by

$$\delta + \exp(X)\pi = \exp(\text{ad}(X))(\delta + \pi)$$

Put

$$M(\mathfrak{g}^\bullet) = MC(\mathfrak{g}^\bullet) / \exp(t\mathfrak{g}^0[[t]]) \quad (2.19)$$

For an L_∞ homomorphism $f : \mathfrak{g}^\bullet \rightarrow \mathfrak{h}^\bullet$, there is a well-defined map

$$M(\mathfrak{g}^\bullet) \rightarrow M(\mathfrak{h}^\bullet)$$

induced by

$$\pi \mapsto \sum_{n=1}^{\infty} \frac{1}{n!} f_n(\pi, \dots, \pi) \quad (2.20)$$

If f is an L_{∞} quasi-isomorphism then the above map is a bijection. Finally, note that $M(\mathfrak{g}_S^{\bullet}(M))$ is the set of equivalence classes of formal Poisson structures on M and $M(\mathfrak{g}_G^{\bullet}(A_0))$ is the set of isomorphism classes of deformations of A_0 for any algebra A_0 . Indeed, the equation $\delta\Pi + \frac{1}{2}[\Pi, \Pi] = 0$ is equivalent to $[m + \Pi, m + \Pi] = 0$, which is equivalent to the product $a * b = ab + \Pi(a, b)$ being associative.

Remark 2.4.2. One can easily define spaces $MC(\mathfrak{g}^{\bullet})$ and $M(\mathfrak{g}^{\bullet})$ for any L_{∞} algebra \mathfrak{g}^{\bullet} . For example, the Maurer-Cartan equation from (2.18) becomes

$$\sum_{n=1}^{\infty} \frac{1}{n!} [\pi, \dots, \pi]_n = 0$$

3. FORMALITY CONJECTURES FOR HOCHSCHILD AND CYCLIC CHAINS

3.1. Hochschild and cyclic chain complexes. For an algebra A over k , define for $n \geq 0$

$$C_n(A, A) = A \otimes \overline{A}^{\otimes n}$$

(recall that $\overline{A} = A/k1$); define $b : C_n(A, A) \rightarrow C_{n-1}(A, A)$ by

$$b(a_0 \otimes \dots \otimes a_n) = (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n \quad (3.1)$$

One has $b^2 = 0$. The complex $(C_{\bullet}(A, A), b)$ is called the Hochschild chain complex of A with coefficients in the bimodule A . The homology of this complex is denoted by $H_{\bullet}(A, A)$, or $HH_{\bullet}(A)$, and is called *the Hochschild homology* of A .

Remark 3.1.1. If $A = C^{\infty}(M)$, one has to use one of the following three definitions of tensor powers of A :

1. $A^{\otimes n+1} = C^{\infty}(M^{n+1})$
2. $A^{\otimes n+1} = \text{germs}_{\Delta} C^{\infty}(M^{n+1})$
3. $A^{\otimes n+1} = \text{jets}_{\Delta} C^{\infty}(M^{n+1})$

where Δ is the diagonal in M^{n+1} . One defines $A \otimes \overline{A}^{\otimes n}$ accordingly. All the definitions above lead to the same answer for the Hochschild cohomology: the map

$$\mu : (C_\bullet(C^\infty(M), C^\infty(M)), b) \rightarrow (\Omega^\bullet(M), 0) \quad (3.2)$$

defined by

$$\mu(a_0 \otimes \cdots \otimes a_n) = \frac{1}{n!} a_0 da_1 \cdots da_n \quad (3.3)$$

is a quasi-isomorphism of complexes (cf. [C]).

For $D \in C^d(A, A)$ and $a_0, \dots, a_n \in A$, define

$$\begin{aligned} L_D(a_0, \dots, a_n) = & \sum_{i=0}^{n-d} (-1)^{(d-1)(i+1)} a_0 \otimes \cdots \otimes a_i \otimes D(a_{i+1}, \dots, a_{i+d}) \otimes \cdots \otimes a_n + \\ & \sum_{j=n-d}^n (-1)^{n(j+1)} D(a_{j+1}, \dots, a_n) \otimes a_{d+j-n} \cdots \otimes a_j \end{aligned} \quad (3.4)$$

One has

$$[L_D, L_E] = L_{[D, E]_G}$$

and

$$b = L_m$$

Thus the operators L_D define an action of the differential graded Lie algebra $\mathfrak{g}_G^\bullet(A)$ on the complex $C_\bullet(A, A)$.

Recall that *the cyclic differential*

$$B : C_n(A, A) \rightarrow C_{n+1}(A, A) \quad (3.5)$$

is defined by

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^n (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_0 \otimes \cdots \otimes a_{i-1} \quad (3.6)$$

One has $b^2 = Bb + bB = B^2 = 0$, as well as

$$[B, L_D] = 0$$

Following Getzler's notation, put

$$CC_\bullet^-(A) = (C_\bullet(A, A)[[u]], b + uB) \quad (3.7)$$

where u is a formal variable of degree -2 . One sees that $CC_\bullet^-(A)$ is a differential graded module over the differential graded algebra $\mathfrak{g}_G^\bullet(A)$ for any algebra A .

Now consider the space $\Omega^\bullet(M)$ of differential forms on a manifold M . For a multivector field $\pi \in \Gamma(M, \wedge^d T)$, put

$$L_\pi = [d, i_\pi] \quad (3.8)$$

where i_π is the contraction

$$i_\pi : \Omega^\bullet(M) \rightarrow \Omega^{\bullet-d}(M) \quad (3.9)$$

It is well known that

$$L_{[\pi, \psi]_S} = [L_\pi, L_\psi] \quad (3.10)$$

Therefore operators L_π define an action of the algebra $\mathfrak{g}_S^\bullet(M)$ on the graded space $\Omega^\bullet(M)$, as well as on the complex $(\Omega^\bullet(M)[[u]], ud)$.

Theorem 3.1.2. (*Connes, [C]*). *The formula*

$$\mu(a_0 \otimes \cdots \otimes a_n) = \frac{1}{n!} a_0 da_1 \cdots da_n$$

defines a functorial $\mathbb{C}[[u]]$ -linear quasi-isomorphism

$$CC_\bullet^-(C^\infty(M)) \rightarrow (\Omega^\bullet(M)[[u]], ud) \quad (3.11)$$

Under the homomorphisms

$$H_\bullet(C^\infty(M), C^\infty(M)) \rightarrow \Omega^\bullet(M) \quad (3.12)$$

$$HC_\bullet^-(C^\infty(M)) \rightarrow H^\bullet(M)[[u]] \quad (3.13)$$

which are induced by μ on cohomology, the operators induced by L_D become L_π where D is a Hochschild cocycle and π is the image of the cohomology class of D under the isomorphism from Theorem 2.2.2.

3.2. L_∞ modules. An L_∞ module over an L_∞ algebra \mathfrak{g}^\bullet is a graded vector space M^\bullet together with a coderivation ∂_M of degree 1 of the free differential graded comodule $S(\mathfrak{g}^\bullet[1]) \otimes M^\bullet$ such that $\partial_M^2 = 0$. A coderivation ∂_M , without the condition $\partial_M^2 = 0$, is uniquely determined by its composition with the projection of $S(\mathfrak{g}^\bullet[1]) \otimes M^\bullet$ onto M^\bullet . This composition is a sequence of linear maps

$$\phi_n : \wedge^n \mathfrak{g}^\bullet \otimes M^\bullet \rightarrow M^\bullet \quad (3.14)$$

of degree $1 - n$. The condition $\partial_M^2 = 0$ is equivalent to the following: for any homogeneous elements x_1, \dots, x_n of \mathfrak{g}^\bullet and m of M^\bullet ,

$$\begin{aligned} & \sum \pm \phi_{p+1}(x_{i_1}, \dots, x_{i_p}, \phi_q(x_{j_1}, \dots, x_{j_q}, m)) + \\ & + \sum \pm \phi_{q+1}([x_{i_1}, \dots, x_{i_p}]_p, x_{j_1}, \dots, x_{j_q}, m) = 0 \end{aligned} \quad (3.15)$$

where the signs are computed and the sum is taken as in (2.14). In particular,

$$\delta_M = \phi_0$$

is a differential on M^\bullet . The maps ϕ_n , $n \geq 0$, define an L_∞ module structure on M^\bullet if and only if the maps ϕ_n , $n \geq 1$, define an L_∞

morphism $\mathfrak{g}^\bullet \rightarrow \text{Hom}(M^\bullet, M^\bullet)$ where the right hand side is viewed as a differential graded Lie algebra with the differential $[\delta_M, ?]$.

A morphism of L_∞ modules over \mathfrak{g}^\bullet , $\varphi : M^\bullet \rightarrow N^\bullet$ is by definition a morphism of differential graded comodules $S(\mathfrak{g}^\bullet[1]) \otimes M^\bullet \rightarrow S(\mathfrak{g}^\bullet[1]) \otimes N^\bullet$. Such a morphism is uniquely determined by maps

$$\varphi_n : \wedge^n \mathfrak{g}^\bullet \otimes M^\bullet \rightarrow N^\bullet$$

of degree $-n$, $n \geq 0$, satisfying

$$\begin{aligned} & \sum \pm \varphi_{q+1}([x_{i_1}, \dots, x_{i_p}]_p, x_{j_1}, \dots, x_{j_q}, m) + \\ & + \sum \pm \varphi_{p+1}(x_{i_1}, \dots, x_{i_p}, \phi_q(x_{j_1}, \dots, x_{j_q}, m)) = \\ & = \sum \pm \phi_{p+1}(x_{i_1}, \dots, x_{i_p}, \varphi_q(x_{j_1}, \dots, x_{j_q}, m)) \end{aligned} \quad (3.16)$$

Remark 3.2.1. An equivalent definition of an L_∞ module structure on M^\bullet is the following. Let \mathfrak{g}^\bullet be an L_∞ algebra and M^\bullet a graded vector space. On the graded space $\mathfrak{g}^\bullet \oplus M^\bullet$ consider another grading in which \mathfrak{g}^\bullet is of degree zero and M^\bullet is of degree one. Consider an L_∞ algebra structure on $\mathfrak{g}^\bullet \oplus M^\bullet$ such that:

1. \mathfrak{g}^\bullet is an L_∞ subalgebra of $\mathfrak{g}^\bullet \oplus M^\bullet$
2. all the operations $[\ , \dots,]_n$ are of degree zero with respect to the second grading on $\mathfrak{g}^\bullet \oplus M^\bullet$
3. $[m_1, m_2, \dots]_n = 0$ for any $m_1, m_2 \in M^\bullet$.

Similarly, one can define a morphism of L_∞ modules as an L_∞ morphism $f : \mathfrak{g}^\bullet \oplus M^\bullet \rightarrow \mathfrak{g}^\bullet \oplus N^\bullet$ which is of degree zero with respect to the second grading and for which $f_n(m_1, m_2, \dots) = 0$ for any $m_1, m_2 \in M^\bullet$.

3.3. Formality conjecture for chains. Let K be the L_∞ quasi-isomorphism from Theorem 2.4.1. Via K , the differential graded Lie algebra $\mathfrak{g}_S^\bullet(M)$ acts on $C_\bullet(C^\infty(M), C^\infty(M))$ and $CC_\bullet^-(C^\infty(M))$ as on L_∞ modules.

Conjecture 3.3.1. *There exists a natural quasi-isomorphism of L_∞ modules*

$$\phi : C_\bullet(C^\infty(M), C^\infty(M)) \rightarrow (\Omega^\bullet(M), 0)$$

such that ϕ_0 is the quasi-isomorphism μ of Connes.

This conjecture extends to the following

Conjecture 3.3.2. *There exists a natural $\mathbb{C}[[u]]$ -linear quasi-isomorphism of L_∞ modules*

$$\phi : CC_\bullet^-(C^\infty(M)) \rightarrow (\Omega^\bullet(M)[[u]], ud)$$

such that ϕ_0 is the Connes quasi-isomorphism μ .

4. HOCHSCHILD AND CYCLIC COMPLEXES OF DEFORMED ALGEBRAS

Let π be a formal Poisson structure on a manifold M . The isomorphism from Theorem 2.1.1 provides a star product $*_\pi$ on $C^\infty(M)$. Put

$$A(\pi) = (C^\infty(M), *_\pi) \quad (4.1)$$

This is an algebra over $k = \mathbb{C}[[t]]$. By $A(\pi)^{\otimes_k(n+1)}$ we will denote $C^\infty(M)^{\otimes(n+1)}[[t]]$ (cf. Remark 3.1.1); similarly for $A(\pi) \otimes \overline{A(\pi)}^{\otimes_k n}$. If Conjecture 3.3.2 is true, then one gets

Corollary 4.0.3. *There exists a quasi-isomorphism*

$$\mu^\pi : C_\bullet(A(\pi), A(\pi)) \rightarrow (\Omega^\bullet(M)[[t]], L_\pi) \quad (4.2)$$

which extends to a $\mathbb{C}[[u]]$ -linear quasi-isomorphism

$$\mu^\pi : CC_\bullet^-(A(\pi)) \rightarrow (\Omega^\bullet(M)[[u]][[t]], L_\pi + ud) \quad (4.3)$$

Proof. Let K be the quasi-isomorphism from Theorem 2.4.1. One checks that the formula

$$\mu^\pi(c) = \sum_{n=0}^{\infty} \frac{1}{n!} K_n(\pi, \dots, \pi, c) \quad (4.4)$$

defines a quasi-isomorphism of complexes. \square

Remark 4.0.4. The complexes in the right hand side of formulas (4.2, 4.3) were studied by Brylinski as quasi-classical approximations to the left hand sides. If $\pi = t\pi_0$ where π_0 is a Poisson structure, then filtration by powers of t defines a spectral sequence with the E^1 term equal to the right hand side; this spectral sequence converges to the left hand side ([Bryl]). The above Corollary implies that this spectral sequence degenerates at E^1 .

Corollary 4.0.5. *The space of continuous $\mathbb{C}[[t]]$ -valued traces on $A(\pi)$ is isomorphic to the space of continuous $\mathbb{C}[[t]]$ -linear $\mathbb{C}[[t]]$ -valued functionals on $C^\infty(M)$ which annihilate all Poisson brackets $\{f, g\}_\pi$*

(compare with [CFS], [Fe], [NT3] for the symplectic case).

4.1. The \hat{A} class of a Poisson manifold. Consider a C^∞ manifold M with a Poisson structure π_0 . Assuming that Conjecture 3.3.2 is true, define the cohomology class $\hat{A}(\pi_0)$ in $H^{\text{ev}}(M, \mathbb{C}[[t]])$ as follows.

Let $\pi = t\pi_0$. Recall that for any k -algebra A the periodic cyclic complex of A is defined by

$$CC_\bullet^{\text{per}}(A) = (C_\bullet(A, A)[u^{-1}, u], b + uB) \quad (4.5)$$

Localizing the L_∞ quasi-isomorphism μ^π from Corollary 4.0.3 with respect to u , one gets an L_∞ quasi-isomorphism

$$\mu^\pi : CC_\bullet^{per}(A(\pi)) \rightarrow (\Omega^\bullet(M)[u^{-1}, u][[t]], L_\pi + ud) \quad (4.6)$$

Now note that the complex in the right hand side of (4.6) is isomorphic to the complex $(\Omega^\bullet(M)[u^{-1}, u][[t]], ud)$. Indeed, $L_\pi = [d, i_\pi]$ and the desired isomorphism is given by $\exp(-u^{-1}ti_\pi)$. One gets a quasi-isomorphism

$$CC_\bullet^{per}(A(\pi)) \rightarrow (\Omega^\bullet(M)[u^{-1}, u][[t]], ud) \quad (4.7)$$

If one views 1 as an element of $C_0(A(\pi), A(\pi))$, and thus of $CC_0^{per}(A(\pi))$, then the value of the quasi-isomorphism (4.7) at 1 is an element of degree zero in $H^\bullet(M, \mathbb{C}[[t]][u^{-1}, u])$, so it can be viewed as an element $\hat{A}(\pi_0)$ of $H^{ev}(M, \mathbb{C}[[t]])$. Conjecturally, this class does not depend on t .

Consider the situation when π_0 is a regular Poisson structure. In this case, the symplectic leaves of π_0 form a foliation \mathcal{F} . The tangent bundle $T\mathcal{F}$ of this foliation is an $Sp(2n)$ -bundle, and one can reduce its structure group to the maximal compact subgroup $U(n)$. Let $\hat{A}(T\mathcal{F})$ be the \hat{A} class of the resulting $U(n)$ -bundle.

More generally, suppose that π_0 comes from a symplectic Lie algebroid (\mathcal{E}, ω) ([MK], [BB]; cf. below for the definitions). This generality was suggested to us by Weinstein.

Conjecture 4.1.1. *If π_0 comes from a symplectic Lie algebroid (\mathcal{E}, ω) then*

$$\hat{A}(\pi_0) = \hat{A}(\mathcal{E})$$

Recall that a *Lie algebroid* is a vector bundle \mathcal{E} whose sections form a sheaf of Lie algebras, together with a morphism of sheaves of Lie algebras (*the anchor map*)

$$\rho : \Gamma(\mathcal{E}) \rightarrow \Gamma(T) \quad (4.8)$$

such that

$$[\xi, f\eta] = L_{\rho(\xi)}(f)\eta + f[\xi, \eta] \quad (4.9)$$

where ξ, η are local sections of \mathcal{E} and f is a local function. Recall that for a Lie algebroid \mathcal{E} , the de Rham complex is defined:

$$\begin{aligned} {}^{\mathcal{E}}\Omega^{\bullet}(M) &= \Gamma(M, \wedge^{\bullet}\mathcal{E}^*) \\ {}^{\mathcal{E}}\Omega^{\bullet}(M) &\xrightarrow{d} {}^{\mathcal{E}}\Omega^{\bullet+1}(M) \\ (d\omega)(\xi_1, \dots, \xi_{m+1}) &= \sum_{i=1}^{m+1} (-1)^{i-1} \rho(\xi_i) \omega(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{m+1}) + \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([\xi_i, \xi_j], \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots) \end{aligned} \quad (4.10)$$

The algebra of \mathcal{E} -differential operators ${}^{\mathcal{E}}D_M$ is the quotient of the enveloping algebra $U(\Gamma(\mathcal{E}))$ by the ideal generated by the elements

$$\xi f \eta - L_{\rho(\xi)}(f) \eta - f \xi \eta$$

where ξ, η are local sections of \mathcal{E} and f is a local function.

A *symplectic Lie algebroid* is a Lie algebroid \mathcal{E} together with a non-degenerate closed \mathcal{E} -form $\omega \in {}^{\mathcal{E}}\Omega^{\bullet}(M)$. We denote by $\pi_0 \in \Gamma(M, \wedge^2\mathcal{E})$ the image of ω under the isomorphism $\wedge^2\mathcal{E}^* \rightarrow \wedge^2\mathcal{E}$ induced by ω . The bivector field $(\wedge^2\rho)(\pi_0)$ is a Poisson structure. We will denote this Poisson structure also by π_0 .

Let us outline the reasoning for which the above conjecture should be true. Corollary 4.0.3 is true for $\pi = t\pi_0$ where π_0 is a regular Poisson structure or, more generally, when π_0 is given by a symplectic Lie algebroid (\mathcal{E}, ω) . The cohomology of the above complex will be denoted by ${}^{\mathcal{E}}H^{\bullet}(M)$. When \mathcal{F} is a foliation with a leafwise symplectic form ω and \mathcal{E} is the tangent bundle of this foliation, then ${}^{\mathcal{E}}\Omega^{\bullet}(M)$ is the de Rham complex of leafwise forms. The anchor map extends to a morphism of complexes $\Omega^{\bullet}(M) \rightarrow {}^{\mathcal{E}}\Omega^{\bullet}(M)$.

In [NT1], we studied star products on $C^{\infty}(M)$ for which the corresponding Poisson structure π_0 is given by a symplectic Lie algebroid (\mathcal{E}, ω) and the operators P_m are \mathcal{E} -bidifferential (call them \mathcal{E} -deformations). We regard two \mathcal{E} -deformations as equivalent if there is an equivalence of star products $T = \text{Id} + \sum_{m=1}^{\infty} t^m T_m$ for which all T_m are \mathcal{E} -differential operators. We showed that Fedosov's methods from [F] are applicable in this situation. In particular, to any \mathcal{E} -deformation one can associate a characteristic class

$$\theta \in \frac{1}{t} \omega + {}^{\mathcal{E}}H^2(M, \mathbb{C})[[t]] \quad (4.11)$$

which defines a bijection between the set of equivalence classes of \mathcal{E} -deformations and $\frac{1}{t} \omega + {}^{\mathcal{E}}H^2(M, \mathbb{C})[[t]]$.

Suppose given an \mathcal{E} -deformation $A = (C^\infty(M), *)$ with the characteristic class θ . In [NT1] and [BNT], we constructed a $\mathbb{C}[[u]]$ -linear trace density map

$$\mu^t : CC_\bullet^-(A) \rightarrow (\mathcal{E}\Omega^{2n-\bullet}(M)((t))[[u]], d) \quad (4.12)$$

whose localization with respect to u provides a $\mathbb{C}[u^{-1}, u]$ -linear morphism

$$\mu^t : CC_\bullet^{per}(A) \rightarrow (\mathcal{E}\Omega^{2n-\bullet}(M)((t))[u^{-1}, u], d) \quad (4.13)$$

We proved

Theorem 4.1.2.

$$\mu^t(1) = \sum_{p \geq 0} \widehat{A}(\mathcal{E})_{2p} u^{n-p}$$

Now assume for simplicity that $\theta = \frac{1}{t}\omega$.

Let $N = (-1)^n t^{k-n}$ on $\mathcal{E}\Omega^k(M)$. Let $* : \mathcal{E}\Omega^\bullet(M) \rightarrow \mathcal{E}\Omega^{2n-\bullet}(M)$ be the symplectic star operator. Consider the sequence of morphisms of complexes

$$\begin{aligned} & (\mathcal{E}\Omega^{2n-\bullet}(M)((t))((u)), d) \xrightarrow{N} (\mathcal{E}\Omega^{2n-\bullet}(M)((t))((u)), td) \\ & \xrightarrow{\exp(ut^{-1}i_{\pi_0})} (\mathcal{E}\Omega^{2n-\bullet}(M)((t))((u)), td + uL_{\pi_0}) \\ & \xrightarrow{*} (\mathcal{E}\Omega^\bullet(M)((t))((u)), tL_{\pi_0} + ud) \\ & \xrightarrow{\exp(-tu^{-1}i_{\pi_0})} (\mathcal{E}\Omega^\bullet(M)((t))((u)), ud) \end{aligned} \quad (4.14)$$

Denote the composition of the above maps by ν .

Conjecture 4.1.3. *The quasi-isomorphism (4.7), composed with*

$$\Omega^\bullet(M)((t))((u)) \rightarrow \mathcal{E}\Omega^\bullet(M)((t))((u)), \quad (4.15)$$

is equal to $\nu\mu^t$.

Compose ν with the isomorphism

$$\mathcal{E}\Omega^\bullet(M)((t))((u)) \rightarrow \mathcal{E}\Omega^\bullet(M)((t))((u))$$

which is equal to u^{n-k} on $\mathcal{E}\Omega^k$.

Remark 4.1.4. Note the symmetry between the formal variables u and t in the above calculations.

Denote the resulting composition by ν_0 . We claim that ν_0 is equal to the operator of multiplication by $\exp(-\frac{\omega}{ut})$. Indeed, one checks that $\nu_0(1)$ is equal to $\exp(-\frac{\omega}{ut})$ (we use the fact that for any z

$$\exp(zi_\pi) \frac{1}{n!} \omega^n = z^n \exp(z^{-1}\omega).$$

But ν_0 is a $C^\infty(M)$ -linear endomorphism of ${}^\mathcal{E}\Omega^{2n-\bullet}(M)((t))((u))$, thus it is the operator of multiplication by $\nu_0(1)$. From Theorem 4.1.2 we see that

$$(\nu_0\mu^t)(1) = u^n \sum \hat{A}(\mathcal{E})_{2p} u^{-p} \quad (4.16)$$

Therefore

$$(\nu_0\mu^t)(1) = \sum \hat{A}(\mathcal{E})_{2p} u^p \quad (4.17)$$

Combining the above formula with Conjecture 4.1.3, one sees that the composition of maps (4.15) and (4.7) evaluated at 1 is equal to $\sum \hat{A}(\mathcal{E})_{2p} u^p$.

5. HOMOTOPY GERSTENHABER ALGEBRAS AND MODULES

In this section, we will outline a possible proof of the conjectures above, as well as their generalizations. This proof will follow the lines of the recent proof of the Kontsevich formality theorem, due to Tamarkin [T].

5.1. Definitions. Recall that a graded space V^\bullet is a *Gerstenhaber algebra* if it is a graded commutative associative algebra, $V^\bullet[1]$ is a graded Lie algebra, and the two operations on V^\bullet satisfy the Leibnitz identity

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c] \quad (5.1)$$

The Hochschild cohomology $H^\bullet(A, A)$ of any associative algebra A is a Gerstenhaber algebra on which the product and the bracket are induced by the cup product and the Gerstenhaber bracket respectively ((2.11), (2.6)).

Let us recall a definition of a G_∞ algebra, or a strong homotopy Gerstenhaber algebra. For a graded vector space V^\bullet , consider the free Lie coalgebra $\text{coLie}(V^\bullet[1])$ and the free cocommutative coalgebra $S(\text{coLie}(V^\bullet[1]))$. The latter graded space has a structure of a Lie coalgebra which is dual to the Berezin-Kirillov-Kostant Lie algebra structure (in the dual language, $S(\text{Lie}(V^\bullet[1]))$ is a Poisson algebra).

A structure of a G_∞ algebra on V^\bullet is by definition a map $\partial : S(\text{coLie}(V^\bullet[1])) \rightarrow S(\text{coLie}(V^\bullet[1]))$ of degree 1 which is a coderivation with respect to both coalgebra structures and such that $\partial^2 = 0$. Any Gerstenhaber algebra is a G_∞ algebra. For any G_∞ algebra V^\bullet , $V^\bullet[1]$ is an L_∞ algebra.

For any G_∞ algebra V^\bullet , one can define the cochain complex of coderivations of $S(\text{coLie}(V^\bullet[1]))$, with the differential $[\partial, ?]$. The cohomology of this complex is denoted by $H^\bullet(V^\bullet, V^\bullet)$. This is a G_∞ analogue of Hochschild cohomology.

Define a G_∞ morphism $f : V^\bullet \rightarrow W^\bullet$ to be a morphism of differential graded Poisson coalgebras $S(\text{coLie}(V^\bullet[1])) \rightarrow S(\text{coLie}(W^\bullet[1]))$. We say that a G_∞ algebra V^\bullet is formal if there is a G_∞ quasi-isomorphism

$$H^\bullet(V^\bullet, V^\bullet) \rightarrow V^\bullet \quad (5.2)$$

A standard argument from homological algebra shows that obstructions to formality of a G_∞ algebra V^\bullet lie in $H^\bullet(V^\bullet, V^\bullet)$.

5.2. Tamarkin's proof. In [T], Tamarkin proves the following

Theorem 5.2.1. *For any associative algebra A , the Hochschild cochain complex $C^\bullet(A, A)$ has a structure of a G_∞ algebra whose underlying L_∞ algebra is $\mathfrak{g}_G^\bullet(A)$*

Theorem 5.2.2. *Let $A = \mathbb{C}[[x_1, \dots, x_n]]$ or $A = C^\infty(\mathbb{R}^n)$. The obstructions to formality of the G_∞ algebra $C^\bullet(A, A)$ are equal to zero.*

This shows that the above algebras are formal as G_∞ algebras. From this, using an argument with Gelfand-Fuks cohomology as in [K], one deduces

Theorem 5.2.3. *Let $A = C^\infty(M)$. Then $C^\bullet(A, A)$ is formal as a G_∞ algebra.*

5.3. Generalized formality conjecture for chains. Conjecture 3.3.1 can be generalized along the lines of the previous subsection as follows. First, one can define G_∞ modules and their homomorphisms as one did in the L_∞ case (following any of the above definitions, for example the one from Remark 3.2.1).

The problem with this definition is that, for example, $\Omega^\bullet(M)$ is not a Gerstenhaber module over the Gerstenhaber algebra $\Gamma(\wedge^\bullet(T))$. To correct this, for any Gerstenhaber algebra V^\bullet define a new Gerstenhaber algebra $V^\bullet[\epsilon]$ by

$$(a + \epsilon b)(c + \epsilon d) = ac + \epsilon(bc + (-1)^{|a|}ad + (-1)^{|a|}[a, c]) \quad (5.3)$$

$$[a + \epsilon b, c + \epsilon d] = [a, c] + \epsilon([b, c] + (-1)^{|a|+1}[a, d]) \quad (5.4)$$

(This is a deformation of V^\bullet with an odd parameter along the Poisson bracket; the specifics of the graded case is that the deformed algebra remains commutative. Note that an isomorphism of this deformation

to the trivial one is precisely a BV operator). Using Tamarkin's methods, one can prove that for any algebra A there is a G_∞ structure on $C_\bullet(A, A)[\epsilon]$ which induces the structure (5.3), (5.4) on $H_\bullet(A, A)[\epsilon]$.

Conjecture 5.3.1. *For any associative algebra A , the Hochschild chain complex $C_\bullet(A, A)$ is a G_∞ module over the G_∞ algebra $C^\bullet(A, A)[\epsilon]$. The underlying structure of an L_∞ module over $C^\bullet(A, A)$ on $C_\bullet(A, A)$ is given by the action of $C^\bullet(A, A)$ by operators L_D (3.4).*

If the above conjecture is true then, by virtue of Theorem 5.2.1, both $C_\bullet(C^\infty(M), C^\infty(M))$ and $\Omega^\bullet(M)$ are G_∞ modules over $\Gamma(M, \wedge^\bullet T)[\epsilon]$.

Conjecture 5.3.2. *There is a quasi-isomorphism of G_∞ modules*

$$C_\bullet(C^\infty(M), C^\infty(M)) \rightarrow \Omega^\bullet(M)$$

To generalize Conjecture 3.3.2, first note that the operator $\frac{\partial}{\partial \epsilon}$ is a BV operator on the algebra $V^\bullet[\epsilon]$ (5.3), (5.4). Conjecturally, in an appropriate sense, $C_\bullet^-(A, A)[\epsilon]$ is a homotopy BV algebra and $CC_\bullet^-(A)[\epsilon]$ is a homotopy BV module over it.

Let us finish with a partial case of the statement before Conjecture 5.3.1 which can be obtained by explicit computation.

Theorem 5.3.3. [DT]. *On the Hochschild chain complex $C_\bullet(A, A)$, there is a structure of an L_∞ module over $\mathfrak{g}_G^\bullet(A)[\epsilon]$ whose restriction to $\mathfrak{g}_G^\bullet(A)$ is given by the operators L_D .*

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